

Mathematical Journal of Okayama University

Volume 48, Issue 1

2006

Article 17

JANUARY 2006

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Abstract

The purpose of this paper is to give a new generalization of the Poisson Kernel in two dimensions and discuss an integral formula for this.

KEYWORDS: Poisson Kernel, Integral Formula

Math. J. Okayama Univ. **48** (2006), 173–180

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ABSTRACT. The purpose of this paper is to give a new generalization of the Poisson Kernel in two dimensions and discuss an integral formula for this.

1. INTRODUCTION

The Poisson Kernel in two dimensions is defined by

$$(1.1) \quad P_r(\theta) = \frac{1 - r^2}{(1 - re^{i\theta})(1 - re^{-i\theta})}.$$

Here r is a real parameter satisfying $|r| < 1$, and $-\infty < \theta < \infty$. It is well-known that $P_r(\theta)$ is periodic in θ with period 2π and the integral formula

$$(1.2) \quad \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta) d\theta = 1$$

holds.

In [2], Haruki and Rassias gave the following new definitions (1.3) and (1.5):

First, they set

$$(1.3) \quad Q(\theta; a, b) \stackrel{\text{def}}{=} \frac{1 - ab}{(1 - ae^{i\theta})(1 - be^{-i\theta})},$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 1. If we take $a = r$ and $b = r$ in (1.3), then we find that (1.3) is a generalization of (1.1).

Afterwards, they proved the following theorem.

Theorem 1.1.

$$(1.4) \quad \frac{1}{2\pi} \int_0^{2\pi} Q(\theta; a, b) d\theta = 1,$$

where a, b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 2. Note that, (1.4) is a generalization of (1.2).

Mathematics Subject Classification. 31A05, 31A10.

Key words and phrases. Poisson Kernel, Integral Formula.

Second, they set

$$(1.5) \quad R(\theta; a, b, c, d) = \frac{L(a, b, c, d)}{(1 - ae^{i\theta})(1 - be^{-i\theta})(1 - ce^{i\theta})(1 - de^{-i\theta})},$$

where a, b, c, d are complex parameters satisfying $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$ and

$$(1.6) \quad L(a, b, c, d) \stackrel{\text{def}}{=} \frac{(1 - ab)(1 - ad)(1 - bc)(1 - cd)}{1 - abcd}.$$

Remark 3. If we take $c = 0$ and $d = 0$ in (1.5), then we find that (1.5) is a generalization of (1.3).

Afterwards, they proved the following theorem.

Theorem 1.2.

$$(1.7) \quad \frac{1}{2\pi} \int_0^{2\pi} R(\theta; a, b, c, d) d\theta = 1,$$

where a, b, c, d are complex parameters satisfying $|a| < 1$, $|b| < 1$, $|c| < 1$ and $|d| < 1$.

In this paper, we shall generalize (1.5) and (1.7).

Set

$$(1.8) \quad S(\theta; x, y, z, t, u, v) = \frac{L(x, y, z, t, u, v)}{(1 - xe^{i\theta})(1 - ye^{-i\theta})(1 - ze^{i\theta})(1 - te^{-i\theta})(1 - ue^{i\theta})(1 - ve^{-i\theta})},$$

where x, y, z, t, u, v are complex parameters satisfying $|x| < 1$, $|y| < 1$, $|z| < 1$, $|t| < 1$, $|u| < 1$, $|v| < 1$ and

$$(1.9) \quad L(x, y, z, t, u, v) \stackrel{\text{def}}{=} \frac{(1 - xy)(1 - xt)(1 - xv)(1 - yz)(1 - yu)(1 - zt)(1 - zv)(1 - tu)(1 - uv)}{K(x, y, z, t, u, v)},$$

where

$$(1.10) \quad \begin{aligned} K(x, y, z, t, u, v) = & 1 - [xz + xu + zu][yt + yv + tv] \\ & + xzu[y^2(t + v) + t^2(y + v) + v^2(y + t)] \\ & + ytv[x^2(z + u) + z^2(x + u) + u^2(x + z)] \\ & - [x^2zu + xz^2u + xzu^2][y^2tv + yt^2v + ytv^2] \\ & + 4xyztuv + x^2y^2z^2t^2u^2v^2. \end{aligned}$$

Remark 4. By taking $u = 0$ and $v = 0$ in (1.8), we find that (1.8) is a generalization of (1.5).

The purpose of this paper is to prove the following

Main Theorem.

$$(1.11) \quad \frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, t, u, v) d\theta = 1,$$

where x, y, z, t, u, v are complex parameters satisfying $|x| < 1, |y| < 1, |z| < 1, |t| < 1, |u| < 1, |v| < 1$.

2. PROOF OF THE MAIN THEOREM

Proof. We get

$$(2.1) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})}$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{(e^{i\theta})^2}{(1-xe^{i\theta})(e^{i\theta}-y)(1-ze^{i\theta})(e^{i\theta}-t)(1-ue^{i\theta})(e^{i\theta}-v)} ie^{i\theta} d\theta.$$

If we substitute $w = e^{i\theta}$, then we have

$$(2.2) \quad ie^{i\theta} d\theta = dw.$$

We set

$$(2.3) \quad f(w) = \frac{w^2}{(1-xw)(w-y)(1-zw)(w-t)(1-uw)(w-v)}.$$

The function $f(w)$ is an analytic function in $|w| \leq 1$ except at $w = y, w = t$ and $w = v$ each of which is a pole of f .

Here there are five cases:

- 1) $y \neq t \neq v$
- 2) $y = t \neq v$
- 3) $y = v \neq t$
- 4) $t = v \neq y$
- 5) $y = t = v$

Case 1. Let $y \neq t \neq v$.

Then, by (2.1), (2.2) and (2.3) we obtain

$$(2.4) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})}$$

$$= \frac{1}{2\pi i} \int_{|w|=1} f(w) dw,$$

where the complex integral of the function $f(w)$ along the unit circle $|w| = 1$ is in the positive direction.

Let R_1 , R_2 and R_3 denote the residues of $f(w)$ at $w = y$, $w = t$ and $w = v$ each of which is a simple pole of f , respectively. So, by the Residue Theorem ([1]) we have

$$(2.5) \quad \frac{1}{2\pi i} \int_{|w|=1} f(w)dw = R_1 + R_2 + R_3.$$

In the following, we shall calculate this residues R_1 , R_2 and R_3 .

We get

$$\begin{aligned} (2.6) \quad R_1 &= \lim_{w \rightarrow y} [(w - y) f(w)] \\ &= \lim_{w \rightarrow y} \frac{w^2}{(1 - xw)(1 - zw)(w - t)(1 - uw)(w - v)} \quad (\text{by (2.3)}) \\ &= \frac{y^2}{(1 - xy)(1 - zy)(y - t)(1 - uy)(y - v)}, \end{aligned}$$

$$\begin{aligned} (2.7) \quad R_2 &= \lim_{w \rightarrow t} [(w - t) f(w)] \\ &= \lim_{w \rightarrow t} \frac{w^2}{(1 - xw)(w - y)(1 - zw)(1 - uw)(w - v)} \quad (\text{by (2.3)}) \\ &= \frac{t^2}{(1 - xt)(t - y)(1 - zt)(1 - ut)(t - v)} \end{aligned}$$

and

$$\begin{aligned} (2.8) \quad R_3 &= \lim_{w \rightarrow v} [(w - v) f(w)] \\ &= \lim_{w \rightarrow v} \frac{w^2}{(1 - xw)(w - y)(1 - zw)(w - t)(1 - uw)} \quad (\text{by (2.3)}) \\ &= \frac{v^2}{(1 - xv)(v - y)(1 - zv)(v - t)(1 - uv)}. \end{aligned}$$

So, from (2.5), (2.6), (2.7) and (2.8) we obtain

$$(2.9) \quad \frac{1}{2\pi i} \int_{|w|=1} f(w)dw = \frac{1}{L(x, y, z, t, u, v)} \quad (\text{by (1.9) and (1.10)}).$$

Therefore, by (1.8), (2.4) and (2.9) we get

$$\frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, t, u, v) d\theta = 1.$$

Case 2. Let $y = t \neq v$.

Hence, (2.3) is of the form

$$(2.10) \quad f(w) = \frac{w^2}{(1-xw)(w-y)^2(1-zw)(1-uw)(w-v)}.$$

Thus, by (2.1), (2.2) and (2.10) we obtain

$$(2.11) \quad \begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})^2(1-ze^{i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})} \\ &= \frac{1}{2\pi i} \int_{|w|=1} f(w)dw, \end{aligned}$$

where the complex integral of the function $f(w)$ along the unit circle $|w| = 1$ is in the positive direction.

Here, note that $f(w)$ is an analytic function in $|w| \leq 1$ except at $w = y$ which is a double pole of f and $w = v$ which is a simple pole of f .

Let R_1 denotes the residue of $f(w)$ at $w = y$ and R_2 denotes the residue of $f(w)$ at $w = v$. By the Residue Theorem, we have

$$(2.12) \quad \frac{1}{2\pi i} \int_{|w|=1} f(w)dw = R_1 + R_2.$$

First, we shall calculate R_1 . By Cauchy's Integral Formula for the derivative ([1]), we have

$$(2.13) \quad \begin{aligned} R_1 &= \frac{1}{2\pi i} \int_{|w|=1} \frac{w^2}{(1-xw)(1-zw)(1-uw)(w-v)} / (w-y)^2 dw \\ &= \left[\frac{d}{dw} \left(\frac{w^2}{(1-xw)(1-zw)(1-uw)(w-v)} \right) \right]_{w=y} \\ &= \frac{y}{(1-xy)(1-zy)(1-uy)(y-v)} \left[2 + \frac{xy}{1-xy} + \frac{zy}{1-zy} + \frac{yu}{1-yu} - \frac{y}{y-v} \right]. \end{aligned}$$

Second, we have

$$(2.14) \quad \begin{aligned} R_2 &= \lim_{w \rightarrow v} [(w-v)f(w)] \\ &= \lim_{w \rightarrow v} \frac{w^2}{(1-xw)(w-y)^2(1-zw)(1-uw)} \end{aligned}$$

$$= \frac{v^2}{(1-xv)(v-y)^2(1-zv)(1-uv)}.$$

So, by (2.12), (2.13) and (2.14) we obtain

$$(2.15) \quad \frac{1}{2\pi i} \int_{|w|=1} f(w)dw = \frac{K(x, y, z, y, u, v)}{(1-xy)^2(1-zy)^2(1-yu)^2(1-xv)(1-uv)(1-zv)}$$

$$= \frac{1}{L(x, y, z, y, u, v)} \quad (\text{by (1.9) and (1.10)}).$$

Thus, by (1.8), (2.11) and (2.15) we have

$$\frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, y, u, v) d\theta = 1.$$

Case 3 and Case 4. The proofs of the integral formulas

$$\frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, t, u, y) d\theta = 1$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, t, u, t) d\theta = 1$$

for Case 3 and Case 4, respectively, are similar to the proof of the Case 2.

Case 5. Let $y = t = v$.

In this case, (2.3) is of the form

$$(2.16) \quad f(w) = \frac{w^2}{(1-xw)(w-y)^3(1-zw)(1-uw)}.$$

Thus, by (2.1), (2.2) and (2.16) we obtain

$$(2.17) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})(1-ze^{i\theta})(1-te^{-i\theta})(1-ue^{i\theta})(1-ve^{-i\theta})}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1-xe^{i\theta})(1-ye^{-i\theta})^3(1-ze^{i\theta})(1-ue^{i\theta})}$$

$$= \frac{1}{2\pi i} \int_{|w|=1} f(w)dw,$$

where the complex integral of the function $f(w)$ along the unit circle $|w| = 1$ is in the positive direction.

So, $f(w)$ is an analytic function in $|w| \leq 1$ except at $w = y$. Note that $f(w)$ has a pole of order 3 at $w = y$. Let R denotes the residue of f at $w = y$. Then, by the Residue Theorem we get

$$(2.18) \quad \frac{1}{2\pi i} \int_{|w|=1} f(w)dw = R.$$

Therefore, we must calculate R . By Cauchy's Integral Formula for the derivative, we have

$$\begin{aligned} (2.19) \quad \frac{1}{2\pi i} \int_{|w|=1} f(w)dw &= \frac{1}{2\pi i} \int_{|w|=1} \frac{w^2}{(1-xw)(1-zw)(1-uw)} / (w-y)^3 dw \\ &= \frac{1}{2!} \left[\frac{d^2}{dw^2} \left(\frac{w^2}{(1-xw)(1-zw)(1-uw)} \right) \right]_{w=y} \\ &= \frac{K(x, y, z, y, u, y)}{(1-xy)^3(1-zy)^3(1-uy)^3} \\ &= \frac{1}{L(x, y, z, y, u, y)} \text{ (by (1.9) and (1.10)).} \end{aligned}$$

Thus, by (1.8), (2.17) and (2.19) we have

$$\frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, y, u, y) d\theta = 1.$$

From the Cases 1, 2, 3, 4 and 5, we get the desired result (1.11). □

Corollary 2.1. *If we set $z = x$, $u = x$ and $t = y$, $v = y$ in the Main Theorem, then we have*

$$(2.20) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-xy)^3}{(1-xe^{i\theta})^3(1-ye^{-i\theta})^3} d\theta = \frac{1+4xy+x^2y^2}{(1-xy)^2}.$$

Proof. By the Main Theorem, we know that

$$1 = \frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, z, t, u, v) d\theta.$$

Then, for $z = x$, $u = x$ and $t = y$, $v = y$ we obtain

$$\begin{aligned}
 1 &= \frac{1}{2\pi} \int_0^{2\pi} S(\theta; x, y, x, y, x, y) d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{L(x, y, x, y, x, y)}{(1 - xe^{i\theta})^3 (1 - ye^{-i\theta})^3} d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - xy)^9}{(1 - xe^{i\theta})^3 (1 - ye^{-i\theta})^3 K(x, y, x, y, x, y)} d\theta \\
 &= \frac{(1 - xy)^6}{K(x, y, x, y, x, y)} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - xy)^3}{(1 - xe^{i\theta})^3 (1 - ye^{-i\theta})^3} d\theta \\
 &= \frac{(1 - xy)^6}{(1 - xy)^4 (1 + 4xy + x^2y^2)} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - xy)^3}{(1 - xe^{i\theta})^3 (1 - ye^{-i\theta})^3} d\theta \\
 &= \frac{(1 - xy)^2}{(1 + 4xy + x^2y^2)} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - xy)^3}{(1 - xe^{i\theta})^3 (1 - ye^{-i\theta})^3} d\theta,
 \end{aligned}$$

where

$$K(x, y, x, y, x, y) = (1 - xy)^4 (1 + 4xy + x^2y^2).$$

Thus, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - xy)^3}{(1 - xe^{i\theta})^3 (1 - ye^{-i\theta})^3} d\theta = \frac{1 + 4xy + x^2y^2}{(1 - xy)^2}.$$

□

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(Received March 11, 2006)